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Complete dynamics of quadratic stochastic quasi non-Volterra operator

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Abstract: This article explores asymptotic cases of trajectories, one of the main problems of Mathematical Biology. The article presents and proves the theorem that the excitable point of a quadratic operator that is not of Volterra type is unique
Keywords: Simpex, internal fixed point, cyclic trinity, ergodic operator, Lyapunov function.

1. Introduction

The notion of a quadratic stochastic operator (QSO) was introduced by S.N. Bernstein [1] when studying some mathematical problems related to the theory of heredity.

We generalize the concept of QSO in a following way. Let (*E, F*) be a measurable space, where *E* is a fixed set and *F* is a σ -algebra of subsets of the set *E*. Let *S*(*E,F*) be the set of all probability measures defined on the measurable space (*E,F*). Further, let $\{P(x,y,A)\colon x,y\in E, A\in F\}$ be a family of functions defined on *E*x*E*x*F* and satisfied the following conditions:

(i) for any fixed *x, y*∈ *E*,
$P(x,y,)\in S(E,F)$ **(0.1)**

(ii) for any fixed measurable set *A*∈*F*, the mapping
$P(,,A)\to R$ **(0.2)**

is a measurable function of two variables.

(iii) for any fixed *x, y*∈*E*, and *A*∈*F*
P(*x*,*y*,*A*)=*P*(*y*,*x*,*A*) **(0.3)**

Condition (0.1) implies that

P(*x*,*y*,*E*)=1 ∀*x*,*y*∈*E* **(0.4)**

Put $V\colon S^{n-1}\to S^{n-1}$ is a quadratic stochastic operator (QSO),

$$S^{n-1}=\left\{\begin{matrix} x=(x_1,x_2,\dots,x_n)\in R^n\colon \\ x_i\geq 0, i=1,2,\dots,n, \sum_{i=1}^{n} x_i=1\end{matrix}\right\}$$

let it be defined in the simplex, namely

$$(Vx)_k=x'_k=\sum_{i,j=1}^{n} P_{ij,k}x_ix_j,\ k=1,2,\dots,n \tag{1}$$

here $P_{ij,k}\geq 0, P_{ij,k}=P_{ji,k}$ and, evidently

$$\sum_{k=1}^{n} P_{ij,k}=1, \forall i,j,k\in\{1,2,\dots,n\}. \tag{2}$$

2. Methods

QSO is called **a strict non-Volterra operator** if the following condition is satisfied:

$$P_{ij,k}=0 \text{for any } k\in\{i,j\} \tag{3}$$

Definition 1. If condition (3) is not satisfied only for $P_{ii,i}$, i.e., $P_{ii,i}>0$, then QSO V defined in S^{n-1} is called **a quasi-strict non-Volterra operator**.

Definition 2. The solution of the equation $W(\lambda)=\lambda$, $\forall\lambda\in S^2$ is called the fixed point of the operator, and the set of fixed points is denoted (defined) in the following way: $Fix(W)=\{\lambda\in S^2\colon W(\lambda)=\lambda\}$.

Definition 3. If the Jacobian of a fixed point λ of the operator W has no eigenvalue in the unit circle, then the point λ is called a **hyperbolic** fixed point.

Definition 3. A hyperbolic fixed point λ is said to be

(i) **attracting** if absolute values of all eigenvalues of the Jacobi matrix $J(\lambda)$ are less than one,

(ii) **repulsive** if absolute values of all eigenvalues of the Jacobi matrix $J(\lambda)$ are greater than one;

(iii) otherwise, it is called a **saddle-type** fixed point.

For each $x^0\in S^{n-1}$, the trajectory $\{x^{(n)}\}, n=0,1,2,\dots$ under the influence of QSO (1) defined as follows:

$$x^{(n+1)}=V(x^{(n)}), n=0,1,2,\dots$$

3. Results and discussion

One of the main problems of mathematical biology is the study of the asymptotic state of trajectories. These problems are completely solved for the QSOs defined by equality (2) and for the Volterra QSOs with the additional condition $P_{ij,k}=0,\forall k\in\{i,j\}$ (see [8]).

In this article, we restrict ourselves to the study of quasi-strictly non-Volterra operators defined in S^2, which have the following form

$$W\colon\begin{cases} x'=P_{11,1}x^2+P_{22,1}y^2+P_{33,1}z^2+2P_{23,1}yz\\ y'=P_{11,2}x^2+P_{22,2}y^2+P_{33,2}z^2+2P_{13,2}xz\\ z'=P_{11,3}x^2+P_{22,3}y^2+P_{33,3}z^2+2P_{12,3}xy\end{cases} \tag{4}$$

where assuming

$$P_{11,1}=\alpha_1, P_{22,1}=\beta_1, P_{33,1}=\gamma_1,$$
$$P_{11,2}=\alpha_2, P_{22,2}=\beta_2, P_{33,2}=\gamma_2,$$
$$P_{11,3}=\alpha_3, P_{22,3}=\beta_3, P_{33,3}=\gamma_3,$$

operator (4) can be represented as:

$$V\colon\begin{cases} x'=\alpha_1x^2+\beta_1y^2+\gamma_1z^2+2yz\\ y'=\alpha_2x^2+\beta_2y^2+\gamma_2z^2+2xz\\ z'=\alpha_3x^2+\beta_3y^2+\gamma_3z^2+2xy\end{cases} \tag{5}$$

where $\alpha_i,\beta_i,\gamma_i\geq 0, i=1,2,3,\quad \sum_{i=1}^{3}\alpha_i=\sum_{i=1}^{3}\beta_i=\sum_{i=1}^{3}\gamma_i=1$.

Assume that the equalities

$$\alpha_1 = \beta_1 = \gamma_1 = a,$$
$$\alpha_2 = \beta_2 = \gamma_2 = b,$$
$$\alpha_3 = \beta_3 = \gamma_3 = c$$

are true. Then operator (5) has the following form:

$$W: \begin{cases} x' = a(x^2+y^2+z^2) + 2yz \\ y' = b(x^2+y^2+z^2) + 2xz \\ z' = c(x^2+y^2+z^2) + 2xy \end{cases} \quad (6)$$

where $a+b+c=1,\ a,b,c \geq 0$.

4. Conclusions

Fixed point of the operator

Theorem 1: If $a \in \left[0;\frac{1}{2}\right]$, $b \in \left[0;\frac{1}{2}\right]$, $c \in [0;1]$, then operator (6) has a single fixed point.

We find the fixed points of operator (6) by solving the system

$$\begin{cases} a(x^2+y^2+z^2) + 2yz = x, \\ b(x^2+y^2+z^2) + 2xz = y, \\ c(x^2+y^2+z^2) + 2xy = z. \end{cases} \quad (7)$$

Using the first equation of (7) and $x = 1-y-z$, compose the equation:

$$2ay^2 + (2(1+a)z + 1 - 2a)y + \\ +(2az^2 + (1-2a)z + a - 1) = 0 \quad (8)$$

Let's solve this quadratic equation with respect to y. Here

$$D_1 = 4(1+2a-3a^2)z^2 + 4(1-3a+2a^2)z + \\ +(1+4a-4a^2)$$

holds for all $a \in [0;1]$ and $z \in [0;1]$, moreover $D_1 > 0$.

For $z \in \left[0;\frac{2a-1+\sqrt{1+4a-4a^2}}{4a}\right]$, the free term of equation (8) will be negative, i.e., $2az^2 + (1-2a)z + a - 1 \leq 0$. And equation (8) has two real solutions with different signs, which look like this:

$$y_1 = \frac{-(2(1+a)z + 1 - 2a) + \sqrt{D_1}}{4a}$$

$$y_2 = \frac{-(2(1+a)z + 1 - 2a) - \sqrt{D_1}}{4a}$$

For arbitrary $a \in [0;1]$ and $z \in \left[0;\frac{2a-1+\sqrt{1+4a-4a^2}}{4a}\right]$, it will be true $0 < y_1 < 1$, $y_2 < 0$.

Since $y \in [0;1]$, we take only y_1.

As above, using the second equation (7) and $y = 1 - x - z$, we compose the equation

$$2bx^2 + (2(1+b)z + 1 - 2b)x + \\ +(2bz^2 + (1-2b)z + b - 1) = 0 \quad (9)$$

Let's solve this quadratic equation with respect to x. Here

$$D_2 = 4(1+2b-3b^2)z^2 + 4(1-3b+2b^2)z + \\ +(1+4b-4b^2)$$

holds for all $b \in [0;1]$ and $z \in [0;1]$, moreover $D_2 > 0$.

When $z \in \left[0;\frac{2b-1+\sqrt{1+4b-4b^2}}{4b}\right]$, the free term of equation (9) will be negative, i.e., $2bz^2 + (1-2b)z + b - 1 \leq 0$. And equation (9) has two real solutions with different signs, which look like this:

$$x_1 = \frac{-(2(1+b)z + 1 - 2b) + \sqrt{D_2}}{4b}$$

$$x_2 = \frac{-(2(1+b)z + 1 - 2b) - \sqrt{D_2}}{4b}$$

For arbitrary $b \in [0;1]$ and $z \in \left[0;\frac{2b-1+\sqrt{1+4b-4b^2}}{4b}\right]$. it will be true $0 < x_1 < 1$, $x_2 < 0$.

Since $x \in [0;1]$, we take only x_1.

Let's substitute the values of x_1 and y_1, respectively, into the equation $x+y+z=1$.

As a result, we get the equality

$$\frac{b}{2(a+b)}\sqrt{D_1} + \frac{a}{2(a+b)}\sqrt{D_2} - \frac{1}{2} = z \quad (10)$$

Introduce in (10) the following notation:

$$f(z) = \frac{b}{2(a+b)}(4(1+2a-3a^2)z^2 + 4(1-3a+ \\ +2a^2)z + (1+4a-4a^2))^{\frac{1}{2}} \\ + \frac{a}{2(a+b)}(4(1+2b-3b^2)z^2 + \\ 4(1-3b+2b^2)z + (1+4b-4b^2))^{\frac{1}{2}} - \frac{1}{2} \quad (11)$$

Now we calculate the derivatives of the first and second order of $f(z)$ with respect to z:

$$f'(z) = \frac{b}{(a+b)}\frac{2(1+2a-3a^2)z+(1-3a+2a^2)}{\sqrt{D_1}} + \\ + \frac{a}{(a+b)}\frac{2(1+2b-3b^2)z+(1-3b+2b^2)}{\sqrt{D_2}}, \quad (12)$$

$$f''(z) = \frac{8ab}{(a+b)}\left(\frac{(1-a)(3-2a^2)}{D_1^{\frac{3}{2}}} + \frac{(1-b)(3-2b^2)}{D_2^{\frac{3}{2}}}\right) + \\ \frac{a}{(a+b)}\frac{2(1+2b-3b^2)z+(1-3b+2b^2)}{\sqrt{D_2}}. \quad (13)$$

For all $a \in \left[0;\frac{1}{2}\right]$ and $b \in \left[0;\frac{1}{2}\right]$, it will be true

$2(1+2a-3a^2)z + (1-3a+2a^2) > 0$,

$2(1+2b-3b^2)z + (1-3b+2b^2) > 0$,

what implies that $f'(z) > 0$.

Also, $f''(z) \geq 0$ holds for all $a \in [0;1]$ and $b \in [0;1]$. It can be seen from here that for arbitrary $a \in \left[0;\frac{1}{2}\right]$ and $b \in \left[0;\frac{1}{2}\right]$, the given function $f(z)$ is increasing and convex.

In addition, the following relations hold for the values of this function at the points 0 and 1:

$$f(0)=\frac{b\sqrt{1+4a-4a^2}+a\sqrt{1+4b-4b^2}}{2(a+b)}-\frac{1}{2}$$

The inequality $1+4a-4a^2\geq 1$ holds for arbitrary $a\in[0;1]$, similarly $1+4b-4b^2\geq 1$ holds for arbitrary $b\in[0;1]$, hence it follows that $f(0)>0$.

When $a=b=\frac{1}{2}$, the function$f(x)$ reaches its maximum value, i.e., $f(0)\leq\frac{\sqrt{2}-1}{2}$. Hence it follows that $0<f(0)\leq\frac{\sqrt{2}-1}{2}$. As above, $f(1)=\frac{b\sqrt{9-8a^2}+a\sqrt{9-8b^2}}{2(a+b)}-\frac{1}{2}$ follows from (6) and $0<f(1)<1$ will be true for arbitrary $a\in[0;1]$, $b\in[0;1]$.

It follows from the above that equation (10) has a unique solution.

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